Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Real Analysis-II

Mid-Semester Examination — February 19, 2024
Maximum marks: 100 — Time: 3 hours
Instructor: B V Rajarama Bhat

In the following a, b are real numbers with $a < b$.

(1) Prove or disprove: Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Then there exists a partition P such that
$$L(P, f) = U(P, f)$$
if and only if f is a constant function. [15]

(2) Define $h : [0, 1] \to \mathbb{R}$ by
$$h(x) = \begin{cases} 1 & \text{if } x = \frac{m}{2^n} \text{ for some } 1 \leq m \leq 2^n, m, n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$
Compute lower and upper Riemann integrals of h. Is h Riemann integrable? [15]

(3) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable iff there exists a sequence of partitions $\{P_n\}_{n \geq 1}$ such that
$$\lim_{n \to \infty} [U(P_n, f) - L(P_n, f)] = 0.$$
[15]

(4) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded monotonic function. Show that f is Riemann integrable. [15]

(5) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded Riemann integrable function. Define $F : [a, b] \to \mathbb{R}$ by
$$F(x) = \int_a^x f(t)dt, \; x \in [a, b].$$
Suppose f is continuous at $c \in [a, b]$. Show that F is differentiable at c and $F'(c) = f(c)$. [15]

(6) Suppose $v : [a, b] \to \mathbb{R}$ is a continuous function and $\int_a^x v(t)dt = 0$ for all rational $x \in [a, b]$. Show that $v(t) = 0$ for all $t \in [a, b]$. [15]

(7) Suppose $g : [a, b] \to \mathbb{R}$ is a continuous function and $\int_a^x g(t)dt = \int_x^b g(t)dt$ for all $x \in [a, b]$. Show that $g(t) = 0$ for all $t \in [a, b]$. [15]